

## Problem 2.23

[Difficulty: 3]

**2.23** Consider the flow field given in Eulerian description by the expression  $\vec{V} = ax\hat{i} + byt\hat{j}$ , where  $a = 0.2 \text{ s}^{-1}$ ,  $b = 0.04 \text{ s}^{-2}$ , and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point  $(x, y) = (1, 1)$  at the instant  $t = 0$ . Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants  $t = 0, 10$ , and  $20 \text{ s}$ .

**Given:** Velocity field

**Find:** Plot of pathline of particle for  $t = 0$  to  $1.5 \text{ s}$  that was at point  $(1, 1)$  at  $t = 0$ ; compare to streamlines through same point at the instants  $t = 0, 1$  and  $1.5 \text{ s}$

**Solution:**

**Governing equations:** For pathlines  $u_p = \frac{dx}{dt}$   $v_p = \frac{dy}{dt}$  For streamlines  $\frac{v}{u} = \frac{dy}{dx}$

**Assumption:** 2D flow

Hence for pathlines  $u_p = \frac{dx}{dt} = a \cdot x$   $a = \frac{1}{5} \frac{1}{\text{s}}$   $v_p = \frac{dy}{dt} = b \cdot y \cdot t$   $b = \frac{1}{25} \frac{1}{\text{s}^2}$

So, separating variables  $\frac{dx}{x} = a \cdot dt$   $dy = b \cdot y \cdot t \cdot dt$   $\frac{dy}{y} = b \cdot t \cdot dt$

Integrating  $\ln\left(\frac{x}{x_0}\right) = a \cdot t$   $x_0 = 1 \text{ m}$   $\ln\left(\frac{y}{y_0}\right) = b \cdot \frac{1}{2} \cdot t^2$   $y_0 = 1 \text{ m}$

Hence  $x(t) = x_0 \cdot e^{a \cdot t}$   $y(t) = y_0 \cdot e^{\frac{1}{2} \cdot b \cdot t^2}$

Using given data  $x(t) = e^{\frac{t}{5}}$   $y(t) = e^{\frac{t^2}{50}}$

For streamlines  $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot t}{a \cdot x}$

So, separating variables  $\frac{dy}{y} = \frac{b \cdot t}{a \cdot x} \cdot dx$  which we can integrate for any given  $t$  ( $t$  is treated as a constant)

Hence  $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot t \cdot \ln\left(\frac{x}{x_0}\right)$

The solution is  $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t}$   $\frac{b}{a} = 0.2$   $x_0 = 1$   $y_0 = 1$

For  $t = 0$   $y = y_0 \cdot \left( \frac{x}{x_0} \right)^{\frac{b}{a} \cdot t} = 1$

$t = 5$   $y = y_0 \cdot \left( \frac{x}{x_0} \right)^{\frac{b}{a} \cdot t} = x$   $\frac{b}{a} \cdot t = 1$

$t = 10$   $y = y_0 \cdot \left( \frac{x}{x_0} \right)^{\frac{b}{a} \cdot t} = x^2$   $\frac{b}{a} \cdot t = 2$

